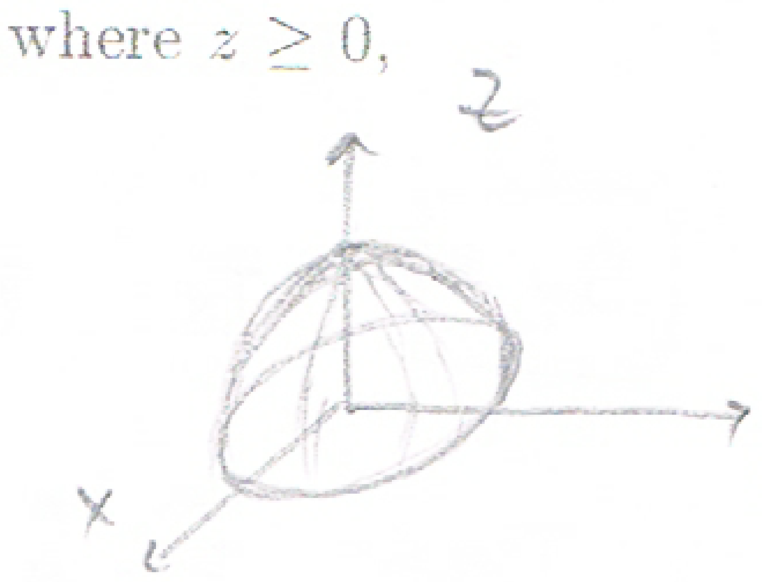
Math 2E Last Quiz Afternoon - June 2nd

Please write your name and ID on the front.

Show all of your work, and simplify all your answers. *There is a question on the back side.

1. Compute $\iint_S z(x^2+y^2)dS$ where S consists of the hemisphere $x^2+y^2+z^2=4$ where $z \geq 0$, and the disk $x^2+y^2 \leq 4$ at $z=0$ (like it is sealing off the hemisphere).

Hint: Break up S into two pieces, the hemisphere and disk.



☆ Let S_1 = Hemisphere, S_2 = disk. $\quad \iint_S = \iint_{S_1} + \iint_{S_2}$.

$\boxed{S_2:}$ On the disk, $z=0$ so $\iint_{S_2} z(x^2+y^2)\,dS = \iint_{S_2} 0\cdot(x^2+y^2)\,dS = \boxed{0}$ +2

$\boxed{S_1}$: For sphere, (radius 2) $\quad x = 2\cos\theta\sin\phi,\ y = 2\sin\theta\sin\phi,\ z = 2\cos\phi$ +1

$\hookrightarrow \vec{r}_\theta = \langle -2\sin\theta\sin\phi,\ 2\cos\theta\cos\phi,\ 0 \rangle$

$\vec{r}_\phi = \langle 2\cos\theta\cos\phi,\ 2\sin\theta\cos\phi,\ -2\sin\phi \rangle$

$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} = -4\cos\theta\sin^2\phi\,\hat{\imath} - 4\sin\theta\sin^2\phi\,\hat{\jmath} - (4\sin^2\theta\sin\phi\cos\phi + 4\cos^2\theta\sin\phi\cos\phi)\,\hat{k}$$

$\hookrightarrow |\vec{r}_\theta \times \vec{r}_\phi| = |-4\langle \cos\theta\sin^2\phi,\ \sin\theta\sin^2\phi,\ \sin\phi\cos\phi\rangle|$

$= 4(\underbrace{\cos^2\theta\sin^4\phi + \sin^2\theta\sin^4\phi}_{\sin^4\phi} + \sin^2\phi\cos^2\phi)^{1/2} = \boxed{4\sin\phi}(\cancel{\sin^2\phi+\cos^2\phi})^{1/2}$ 1

ie. This is part of dS_{sphere}.

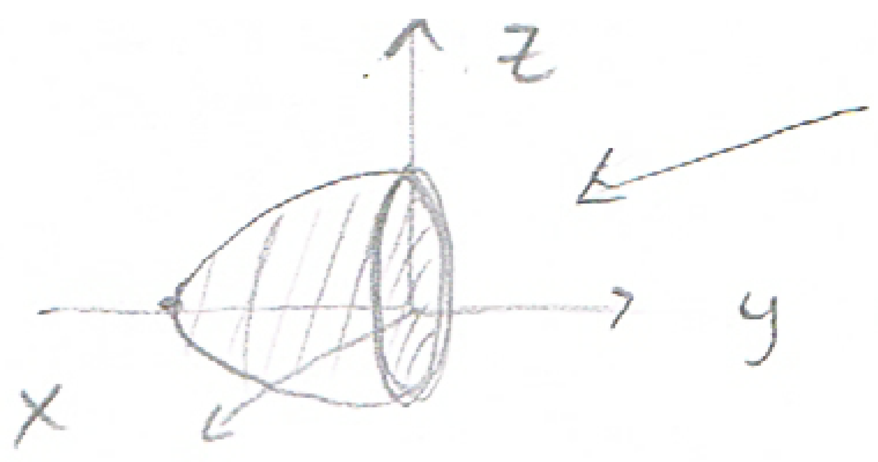
(pull out $\sqrt{\sin^2\phi}$ common multiple)

$\hookrightarrow \iint_{S_1} z(x^2+y^2)\,dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2} 2\cos\phi \cdot 4\sin^2\phi \cdot 4\sin\phi\,d\phi\,d\theta$ +2 +2

θ-indep $\stackrel{}{=} 32\cdot 2\pi \cdot \int_0^{\pi/2} \cos\phi\sin^3\phi\,d\phi \qquad \boxed{\begin{aligned} u &= \sin\phi \\ du &= \cos\phi\,d\phi \end{aligned}}$ +1

$= 64\pi \cdot \int_0^1 u^3\,du = 16\pi \cdot u^4\Big|_0^1 = \boxed{16\pi}$ +1

Hence, $\iint_S = \iint_{S_1} + \iint_{S_2} = 0 + 16\pi = \boxed{16\pi}$ +1



The shadow of the paraboloid in XZ plane is $x^2+z^2 \le 1$.

2. Compute $\iint_S \mathbf{F}\cdot d\mathbf{S}$ where $\mathbf{F}(x,y,z) = < 0,\ y,\ -z >$ and S consists of the paraboloid $y = x^2+z^2-1$ from $-1 \le y \le 0$, and the disk $x^2+z^2 \le 1$ at $y=0$ (like sealing off the paraboloid). Assume positive orientation - the normal points outwards.
Hint: Notice S is the graph of two functions - split S into the paraboloid and the disk. It may help with some computations.

S_1 S_2

$\boxed{S_2}$: The disk is the graph of $y=0$ so, $\vec{n} = <0,1,0>$, already unitized! $= \hat{n}$. +1

Then $\vec{F}\cdot\hat{n} = <0,y,-z>\cdot<0,1,0> = y$ +1

$\hookrightarrow \iint_{S_2} \vec{F}\cdot\hat{n}\,dS = \iint_{S_2} y\,dS = \iint_{S_2} 0\,dS = \boxed{0}$ because we're on $y=0$ plane. +1

$\boxed{S_1}$: Again is a graph, so we can use that for $y = g(x,z)$, for graphs, $\vec{r}_z \times \vec{r}_x = <-g_x, 1, -g_z>$.

Check: At $(0,-1,0)$, the outwards normal points in $-\hat{j}$ direction, not $+\hat{j}$ so we actually need $\vec{r}_x \times \vec{r}_z = <g_x, -1, g_z>$

Here, $\vec{r}_x \times \vec{r}_z = <2x, -1, 2z>$, +2 and then

$\vec{F}(x, x^2+z^2-1, z)\cdot \vec{r}_x\times\vec{r}_z = 0 + (1-x^2-z^2) - 2z^2 = 1-x^2-3z^2$. +1

so we have $\iint_{x^2+z^2\le 1} (1-x^2-3z^2)\,dx\,dz = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (1-r^2\sin^2\theta - 3r^2\cos^2\theta)\,r\,dr\,d\theta$ +1

After the r-integral $= \int_0^{2\pi}\left(\frac{1}{2} - \frac{\sin^2\theta}{4} - \frac{3\cos^2\theta}{4}\right)d\theta = \pi - \int_0^{2\pi}\left[\frac{1}{8}(1-\cancel{\cos 2\theta}) + \frac{3}{8}(1+\cancel{\cos 2\theta})\right]d\theta$

0 by periodicity

$= \pi - 2\pi\left(\frac{1}{8}+\frac{3}{8}\right) = \pi - \pi = \boxed{0}$, so $\iint_S = \iint_{S_1} + \iint_{S_2}$ +1

+2

$= 0 + 0 = \boxed{0}$